
7.2 Review of Matrices

Introduction

A **vector** is an ordered list of objects (for our class, real or complex numbers). Vectors are used in science to describe things like spacial coordinates (in physics) or life stages (in biology). We will use vectors to write systems of ODE's in a concise manner.

Here is an example of a row vector:

```
In[1]:= rv = Table[vj, {j, 1, 4}]
```

```
Out[1]= {v1, v2, v3, v4}
```

and a column vector:

```
In[2]:= cv = Table[wj, {j, 1, 4}];
```

```
MatrixForm[cv]
```

```
Out[3]/MatrixForm=
```

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

A **matrix** is a two dimensional extension of the concept of a vector. A matrix has rows and columns.

Here is an example of a 3×4 matrix

```
In[4]:= A = Table[Table[aj,k, {k, 1, 4}], {j, 1, 3}];
```

```
MatrixForm[A]
```

```
Out[5]/MatrixForm=
```

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix}$$

It has 3 rows and 4 columns. A **square matrix** has the same number of rows as columns. The matrix given above is not square because it has more columns, then it has rows. The **dimension of a matrix** is $m \times n$, where m is the number of rows and n is the number of columns. As stated, the above matrix is 3×4 . A square matrix would be $n \times n$ dimensional.

Vectors can be thought of as matrices, where a row vector has dimension $1 \times n$ and a column vector has dimension $m \times 1$.

Algebra of Matrices

Scalar Multiplication and Addition of Matrices

We can multiply a matrix by a **scalar** (a real or complex number). For example:

```
In[6]:= MatrixForm[c * A]
```

```
Out[6]//MatrixForm=
```

$$\begin{pmatrix} c a_{1,1} & c a_{1,2} & c a_{1,3} & c a_{1,4} \\ c a_{2,1} & c a_{2,2} & c a_{2,3} & c a_{2,4} \\ c a_{3,1} & c a_{3,2} & c a_{3,3} & c a_{3,4} \end{pmatrix}$$

Note that every element of the matrix is multiplied by the scalar. If two matrices have the same dimensions, then we can add them together (component wise). For example: If

```
In[7]:= B = Table[Table[bj,k, {k, 1, 4}], {j, 1, 3}];
MatrixForm[B]
```

```
Out[8]//MatrixForm=
```

$$\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \end{pmatrix}$$

then

```
In[9]:= MatrixForm[A + B]
```

```
Out[9]//MatrixForm=
```

$$\begin{pmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & a_{1,3} + b_{1,3} & a_{1,4} + b_{1,4} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & a_{2,3} + b_{2,3} & a_{2,4} + b_{2,4} \\ a_{3,1} + b_{3,1} & a_{3,2} + b_{3,2} & a_{3,3} + b_{3,3} & a_{3,4} + b_{3,4} \end{pmatrix}$$

The **Transpose** of a matrix: if A is an $m \times n$ matrix with elements $a_{j,k}$ (meaning the number in the j row and k column), then the transpose of A , denoted as A^T is an $n \times m$ matrix with elements $a_{k,j}$. For example:

```
In[10]:= MatrixForm[Transpose[A]]
```

```
Out[10]//MatrixForm=
```

$$\begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,2} \\ a_{1,3} & a_{2,3} & a_{3,3} \\ a_{1,4} & a_{2,4} & a_{3,4} \end{pmatrix}$$

The first row of A becomes the first column of A^T , the second row of A becomes the second column of A^T , and so on.

Matrix Multiplication

Multiplying two matrices together is a little complicated. First let's talk about the dot product of two vectors. If two vectors are the same length (dimension), then their **dot** product is the sum of the product of the corresponding elements:

$$(v_1 v_2 v_3 \cdots v_n) \circ (w_1 w_2 w_3 \cdots w_n) = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

In vector notation this is:

$$\vec{v} \circ \vec{w} = \sum_{k=1}^n v_k w_k.$$

Now for matrix Multiplication. If A is an $m \times l$ matrix and B is an $l \times n$ matrix, then AB will be an $m \times n$ matrix whose element are given by $(AB)_{j,k}$ equals (the j^{th} row of A) \circ (the k^{th} column of B).

Note that matrix multiplication is not commutative. If A is 2×4 , B is 4×2 and C is 4×5 , then AB is a 2×2 matrix while BA is a 4×4 matrix. Also AC is a 2×5 , while CA does not exist (you can not multiply a 4×5

and a 2×4 matrix in that order) because the “inner” dimensions do not match. **Two matrices are equal** if they have the same dimension and the same elements ($A = B$ if and only if $a_{j,k} = b_{j,k}$ for all possible values of j and k).

The identity matrix, denoted as I , is the matrix with property that $AI = IA = A$, for any square matrix A . Here are the 2×2 , 3×3 and 4×4 identity matrices:

```
In[11]:= {IdentityMatrix[2] // MatrixForm,
          IdentityMatrix[3] // MatrixForm,
          IdentityMatrix[4] // MatrixForm}

Out[11]= { ( 1 0 ), ( 1 0 0 ), ( 1 0 0 0 )
           ( 0 1 ), ( 0 1 0 ), ( 0 1 0 0 )
           ( 0 0 1 ), ( 0 0 1 0 )
           ( 0 0 0 1 ) }
```

They have one's on the **main diagonal** and zero's everywhere else.

Example:

```
In[12]:= Clear[A, B]

In[13]:= A = {{1, 5, 3, 4}, {2, -1, -2, 1}};
          MatrixForm[A]
```

```
Out[14]//MatrixForm=
  ( 1 5 3 4 )
  ( 2 -1 -2 1 )
```

```
In[15]:= B = {{3, -1, 0, 2}, {1, 0, 1, 0}};
          MatrixForm[B]
```

```
Out[16]//MatrixForm=
  ( 3 -1 0 2 )
  ( 1 0 1 0 )
```

```
In[17]:= M = A.B; (* This is multiplying A and B *)
          MatrixForm[M]
```

 **Dot:** Tensors {{1, 5, 3, 4}, {2, -1, -2, 1}} and {{3, -1, 0, 2}, {1, 0, 1, 0}} have incompatible shapes.

```
Out[18]//MatrixForm=
  {{1, 5, 3, 4}, {2, -1, -2, 1}}.{{3, -1, 0, 2}, {1, 0, 1, 0}}
```

Note that we tried to multiply a 2×4 times a 2×4 and got an error.

```
In[19]:= M = A.Transpose[B]; (* This is multiplying A and B^T *)
          MatrixForm[M]
```

```
Out[20]//MatrixForm=
  ( 6 4 )
  ( 9 0 )
```

```
In[21]:= M.IdentityMatrix[2] // MatrixForm
```

```
Out[21]//MatrixForm=
  ( 6 4 )
  ( 9 0 )
```

```
In[22]:= IdentityMatrix[2].M // MatrixForm
```

```
Out[22]//MatrixForm=
```

$$\begin{pmatrix} 6 & 4 \\ 9 & 0 \end{pmatrix}$$

```
In[23]:= bob = Transpose[B].A;
```

```
MatrixForm[bob]
```

```
Out[24]//MatrixForm=
```

$$\begin{pmatrix} 5 & 14 & 7 & 13 \\ -1 & -5 & -3 & -4 \\ 2 & -1 & -2 & 1 \\ 2 & 10 & 6 & 8 \end{pmatrix}$$

Note that AB^T was a 2×2 matrix while $B^T A$ was a 4×4 matrix, so $AB^T \neq B^T A$. Again, Matrix multiplication is NOT, in general, commutative.

Matrix Inverses

The **inverse** of a square matrix A , denoted A^{-1} , is a square matrix with the property: $AA^{-1} = A^{-1}A = I$, if it exists. The process for finding an inverse of a matrix is outlined in the book on pages 372-374. Here is an example of how Mathematica can be used to find inverses, by using the **Inverse** command and the **RowReduce** command.

```
In[25]:= Clear[A]
```

```
In[26]:= A = {{3, 0, 1}, {0, 2, 4}, {1, 1, 1}};
```

```
MatrixForm[A]
```

```
Out[27]//MatrixForm=
```

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

We can **augment** the matrix on the right with the identity matrix and use **RowReduce**:

```
In[28]:= AAug = {{3, 0, 1, 1, 0, 0}, {0, 2, 4, 0, 1, 0}, {1, 1, 1, 0, 0, 1}};
```

```
MatrixForm[AAug]
```

```
RowReduce[AAug] // MatrixForm
```

```
Out[29]//MatrixForm=
```

$$\begin{pmatrix} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

```
Out[30]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{4} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{3}{8} & -\frac{3}{4} \end{pmatrix}$$

The inverse is the 3×3 matrix given by the last three columns. Notice that we get the identity matrix on the left and the inverse on the right.

Using the Inverse command we get:

```
In[31]:= Inverse[A] // MatrixForm
```

```
Out[31]//MatrixForm=
```

$$\begin{pmatrix} \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} & \frac{3}{2} \\ \frac{1}{4} & \frac{3}{8} & -\frac{3}{4} \end{pmatrix}$$

Now an example of a matrix that does not have an inverse:

```
In[32]:= Clear[bob]
```

```
In[33]:= bob = {{1, 2, 3, 4}, {5, 6, 7, 8}, {9, 10, 11, 12}, {13, 14, 15, 16}};
```

```
MatrixForm[bob]
```

```
bobinverse = Inverse[bob]
```

```
Out[34]//MatrixForm=
```

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

 **Inverse:** Matrix {{1, 2, 3, 4}, {5, 6, 7, 8}, {9, 10, 11, 12}, {13, 14, 15, 16}} is singular.

```
Out[35]= Inverse[{{1, 2, 3, 4}, {5, 6, 7, 8}, {9, 10, 11, 12}, {13, 14, 15, 16}}]
```

Lets see what happens with the augmented matrix:

```
In[36]:= bobaug = {{1, 2, 3, 4, 1, 0, 0, 0}, {5, 6, 7, 8, 0, 1, 0, 0},
```

```
{9, 10, 11, 12, 0, 0, 1, 0}, {13, 14, 15, 16, 0, 0, 0, 1}};
```

```
MatrixForm[bobaug]
```

```
RowReduce[bobaug] // MatrixForm
```

```
Out[37]//MatrixForm=
```

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 5 & 6 & 7 & 8 & 0 & 1 & 0 & 0 \\ 9 & 10 & 11 & 12 & 0 & 0 & 1 & 0 \\ 13 & 14 & 15 & 16 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
Out[38]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & -1 & -2 & 0 & 0 & -\frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 2 & 3 & 0 & 0 & \frac{13}{4} & -\frac{9}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

Notice that we did not get the identity on the left.

Final Thoughts

We can also consider matrices (or vectors) with functions as elements. In this case, we can integrate or differentiate a matrix. See page 375 of the book. The goal here can be demonstrated with the following.

Consider the system of differential equations:

$$u_1' = \cos[t] u_1 + e^t u_2,$$

$$u_2' = e^{-t} u_1 - \sin[t] u_2.$$

If we let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, then the system can be written as a matrix equation:

$$\vec{u}' = \begin{pmatrix} \cos[t] & e^t \\ e^{-t} & -\sin[t] \end{pmatrix} \vec{u}.$$

We can then use theory from matrix and linear algebra to try to solve the system.