7.4 Basic Theory of Systems of First Order Linear Equations

This is intended as a supplement to the material in section 7.4 of the book. Please look through that material before proceeding.

The Notation

We are primarily interested in Linear, Homogeneous, First Order Systems of equations. We can write such a system as

1. $$\vec{x}\,' = P(t)\,\vec{x},$$

where $P(t)$ is a coefficient matrix whose elements could be functions. To indicate multiple solution of this equation we use superscripts instead of different letters:

$$\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}.$$

Note that for a vector, subscripts indicate the component. So a vector in 3D would look like

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The second component of the vector $\vec{x}^{(4)}$ would be $x_2^{(4)}$.

The Theorems

This section is very much like the corresponding section(s) back in chapter 3 that talked about existence, uniqueness and superposition.

Theorem 7.4.1 If the vector functions $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are solutions of the system (1.), then the linear combination $c_1\,\vec{x}^{(1)} + c_2\,\vec{x}^{(2)}$ is also a solution for any constants c_1 and c_2.

The example that they give in the book comes from the equation

$$\vec{x}\,' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{x} \iff x_1\,' = x_1 + x_2 \text{ and } x_2\,' = 4\,x_1 + x_2.$$

We will explore how you find solution to this type of problem in section 7.5, but note that the eigenvalues and eigenvectors for the coefficient matrix are:

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In[1]:= A = {{1, 1}, {4, 1}};
        Eigensystem[A]
Out[2]= {{3, -1}, {{1, 2}, {-1, 2}}}
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So two solutions to the example are

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3\,t} \text{ and } \vec{x}^{(2)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t}.$$

Thus, the family of solutions to this homogeneous system are

$$\vec{x}(t) = c_1 e^{3t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}-1\\2\end{pmatrix}.$$

Recall that a set of vectors is linearly independent provided

$$c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + \ldots + c_n \vec{x}^{(n)} = \vec{0} \Leftrightarrow c_1 = c_2 = \cdots = c_n = 0.$$

Theorem 7.4.2 If the vector functions $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are linearly independent solutions of the system (1.) for each point in the interval $a < t < b$, then each solution $\vec{x} = \vec{\phi}(t)$ of system (1.) can be expressed as a linear combination of $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$:

$$\vec{\phi}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) + \ldots + c_n \vec{x}^{(n)}(t)$$

in exactly one way.

This theorem says that, provided the vector solutions are linearly independent, each solution can be expressed in only one (unique) way. This means that the vectors form a **fundamental set of solutions** for that interval and the family of solutions given by all linear combinations of the vectors is commonly called the **general solution** to equation (1.).

Theorem 7.4.3 If $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are solutions of equation (1.) on the interval $a < t < b$, then in this interval $W\left[\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}\right]$ is either identically zero or else never zero.

Here $W\left[\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}\right](t) = c\,\mathrm{Exp}\left[\int (p_{1,1}(t) + p_{2,2}(t) + \cdots + p_{n,n}(t))\, dt\right]$ is the Wronskian of the system (1.), where $p_{k.k}(t)$ is the k^{th} diagonal element of the matrix $P(t)$.

The following vectors in n-dimensions are called the unit basis vectors:

$$\vec{e}^{(1)} = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix},\ \vec{e}^{(2)} = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \ldots,\ \vec{e}^{(n)} = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$$

Theorem 7.4.4 Let $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ be the solutions of system (1.) that satisfy the initial conditions

$$\vec{x}^{(1)}(t_0) = \vec{e}^{(1)}, \ \ldots, \ \vec{x}^{(n)}(t_0) = \vec{e}^{(n)},$$

respectively, where t_0 is any point in $a < t < b$. Then $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ form a fundamental set of solutions of the system (1.).

This theorem basically says that if the vector solutions to system (1.) are linearly independent at one point in the interval $a < t < b$, then they form a fundamental set of solutions.

Theorem 7.4.5 Consider the system (1.)

$$\vec{x}\,' = P(t)\,\vec{x},$$

where each element of the matrix P is a real-valued continuous function. If $\vec{x} = \vec{u}(t) + i\,\vec{v}(t)$ is a complex-valued solution of equation (1.), then its real part $\vec{u}(t)$ and its imaginary part $\vec{v}(t)$ are also solutions of this equation.

This theorem allows us to deal with complex roots of the characteristic equation, just like in chapter 3.