7.5 Homogeneous, Linear Systems with Constant Coefficients

Example (i)

Consider the system of differential equations:

$$\vec{x}\,' = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \vec{x}.$$

This system of ODE's is equivalent to the two equations $x_1{}' = 2\,x_1$ and $x_2{}' = -x_2$. Since neither of the derivatives depend on the other variable, this is called an uncoupled system. We can solve each of these equation using techniques from chapter 2: $x_1(t) = c_1\,e^{2\,t}$ and $x_2(t) = c_2\,e^{-t}$. If we write the solution in a vector form, we have

$$\vec{x}(t) = c_1\,e^{2\,t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\,e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that the eigenvalues and corresponding eigenvectors for the coefficient matrix are:

```
In[1]:= Eigensystem[{{2, 0}, {0, -1}}]
Out[1]= {{2, -1}, {{1, 0}, {0, 1}}}
```

In this example, the fundamental solutions each have a form of $e^{\lambda\,t}\,\vec{v}$, where $(\lambda, \vec{v})$ are an eigenpair for the coefficient matrix.

Example (ii)

Consider the system of differential equations:

$$\vec{y}\,' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \vec{y}.$$

This system is equivalent to the equations $y_1{}' = 2\,y_1 + 2\,y_2$ and $y_2{}' = y_1 + 3\,y_2$. In this system, since one or more of the derivatives depends on other dependent variables, we say the system is coupled. We can not use techniques out of chapter 2 to solve these equations, but we can note that if $\vec{y}(t) = e^{\lambda\,t}\,\vec{v}$ is a solution, then

$$\lambda\,e^{\lambda\,t}\,\vec{v} = \vec{y}\,' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\vec{y} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} e^{\lambda\,t}\,\vec{v} \Leftrightarrow \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\vec{v} = \lambda\,\vec{v}.$$

This mean that $(\lambda, \vec{v})$ must be an eigenpair of the coefficient matrix. In this case the eigenvalues and corresponding eigenvectors of the coefficient matrix are:

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In[2]:= Eigensystem[{{2, 2}, {1, 3}}]
Out[2]= {{4, 1}, {{1, 1}, {-2, 1}}}
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So the general solution to this system is

$$\vec{y}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Final Thoughts

For distinct, real eigenvalues, the solution to the system is as above. Sections 7.6-7.8 cover the other cases: for complex eigenvalues and eigenvectors, we use the theorem from 7.4 and find the real and imaginary parts, for repeated eigenvalues it is a bit more complicated (sections 7.7 and 7.8). We will end the course with section 7.5: real, distinct eigenvalues.