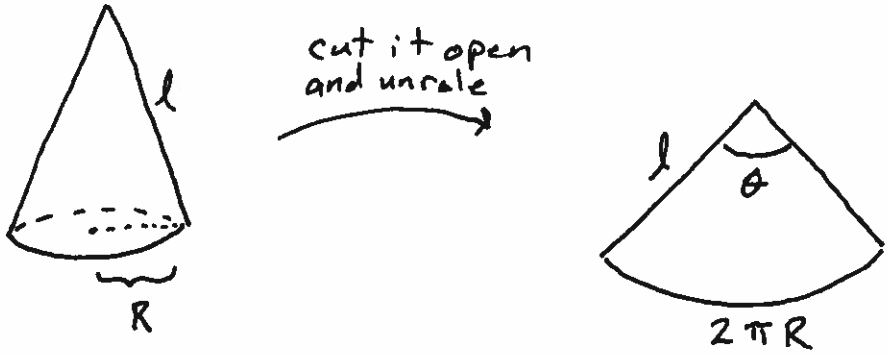
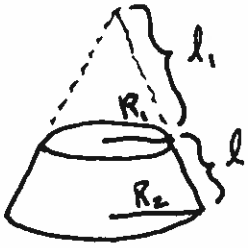
§8.2 Area of Surfaces of Revolution



$$\theta = \frac{2\pi R}{\ell}$$

$$A = \frac{1}{2}\ell^2\theta = \pi R \ell$$



$$A = \pi R_2(\ell_1 + \ell) - \pi R_1 \ell_1$$
$$= \pi\left[(R_2 - R_1)\ell_1 + R_2 \ell\right]$$

But

$$\frac{\ell_1}{R_1} = \frac{\ell_1 + \ell}{R_2} \Rightarrow$$

$$R_2 \ell_1 = R_1 \ell_1 + R_1 \ell \Rightarrow$$

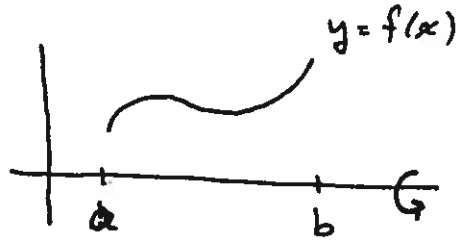
$$(R_2 - R_1)\ell_1 = R_1 \ell$$

So

$$A = \pi(R_1 \ell + R_2 \ell)$$
$$= 2\pi \ell \cdot \frac{R_1 + R_2}{2}$$
$$= 2\pi \ell R$$ where R is the average radius

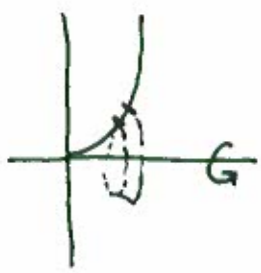
$$\boxed{dS = 2\pi R\, ds}$$

and the total surface area of revolution, S, is



$$S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$

<u>Example (i)</u> $y = x^2$, $x \in [0,2]$, about the x-axis.



$$R = x^2$$

$$ds = \sqrt{1+(2x)^2}\,dx$$

So surface area is

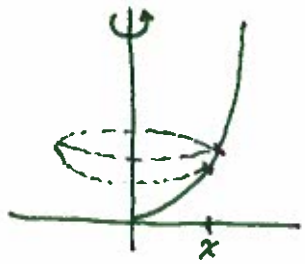
$$S = 2\pi \int_0^2 x^2\sqrt{1+4x^2}\,dx$$

$$\overset{\text{CAS}}{=} \frac{\pi}{32}\left(132\sqrt{17} - \text{ArcSinh}[4]\right)$$

$$\approx 53.226$$

Example (ii)

$y = x^2$, $x \in [0,2]$, about the y-axis.



$R = x$

$ds = \sqrt{1+4x^2}\,dx$

So the surface area is

$$S = 2\pi \int_0^2 x\sqrt{1+4x^2}\,dx$$

simple sub.
$u = 1+4x^2$
$du = 8x\,dx$

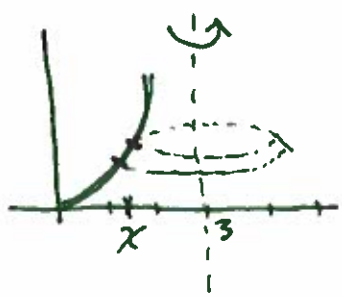
$$= \frac{\pi}{4}\int_1^{17} \sqrt{u}\,du$$

$$= \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^{17}$$

$$= \frac{\pi}{6}\left(17^{3/2} - 1\right)$$

Example (iii)

$y = x^2$, $x \in [0,2]$, about $x = 3$.



$R = 3 - x$

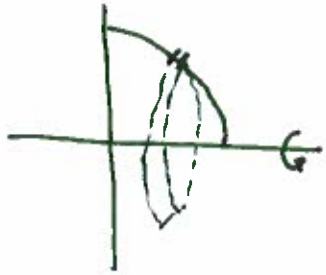
$ds = \sqrt{1-4x^2}\,dx$

$$S = 2\pi \int_0^2 (3-x)\sqrt{1-4x^2}\,dx$$

$$\overset{CAS}{=} \frac{\pi}{6}\left(1 + 19\sqrt{17} + 9\,\text{ArcSinh}[4]\right)$$

$$\approx 51.413$$

Example (iv) The surface area of a sphere of radius R.



look at $y = \sqrt{R^2 - x^2}$, $0 \le x \le R$ rotated about the x-axis to get half of the surface area

Radius of rotation is $\sqrt{R^2 - x^2}$

$$ds = \sqrt{1 + \left(\frac{-x}{\sqrt{R^2 - x^2}}\right)^2}\, dx = \frac{R}{\sqrt{R^2 - x^2}}\, dx$$

So

$$S = 4\pi \int_0^R R\, dx$$

$$= 4\pi R^2$$

Note the pattern:

Circles	Spheres
$A = \pi R^2$	$V = \frac{4}{3}\pi R^3$
$P = 2\pi R$	$S = 4\pi R^2$