Eigen Values and Eigen Vectors

<u>Def</u> If $A\vec{v} = \lambda\vec{v}$, then λ is called an eigenvalue of A and $\vec{v}$ is called a corresponding eigenvector of A.

Note: $A\vec{v} = \lambda\vec{v} \iff (A - \lambda I)\vec{v} = \vec{0}$

If $\vec{v} \neq \vec{0}$, then $\det(A - \lambda I) = 0$

$\det(A - \lambda I) = 0$ is called the characteristic equation for the square matrix A.

<u>Example 1</u> Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$$

E-values:

$$\det(A - \lambda I) = 0 \iff \det\left(\begin{bmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{bmatrix}\right) = 0$$

$$\iff (-4-\lambda)(5-\lambda) - (3)(-6) = 0$$

$$\iff -20 + 4\lambda - 5\lambda + \lambda^2 + 18 = 0$$

$$\iff \lambda^2 - \lambda - 2 = 0$$

$$\iff (\lambda - 2)(\lambda + 1) = 0$$

$$\iff \lambda = -1 \text{ or } \lambda = 2$$

Next the E-vectors:

For $\lambda = -1$: $\quad (A - \lambda I)\vec{v} = \vec{0} \iff \begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is a system of equation in v_1 and v_2 which is dependent. That is, the equations are the same:

$$3v_1 + 6v_2 = 0$$

$$\iff \quad v_1 = -2v_2. \text{ Let } v_2 = t \text{, then } v_1 = -2t.$$

So $\quad \vec{v} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Thus, one of the E-pair is $\left\{-1, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

For $\lambda = 2$: $\quad (A - \lambda I)\vec{v} = \vec{0} \iff \begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Longrightarrow \quad v_1 + v_2 = 0$$

So $\quad \vec{v} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Thus, the other E-pair is $\left\{2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

Mathematica would give us

$$\{\{2, -1\}, \{\{-1, 1\}, \{-2, 1\}\}\}.$$

Note that any scalar multiple of an e-vector is again an e-vector. Sometimes the e-vectors are scaled to be unit vectors.

Example 2

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

First the characteristic equation:

$$\det(A - \lambda I) = 0$$

$$\iff \det\left(\begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}\right) = 0$$

$$\iff (5-\lambda)\det\left(\begin{bmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}\right) - (4)\det\left(\begin{bmatrix} 4 & 2 \\ 2 & 2-\lambda \end{bmatrix}\right)$$

$$+ (2)\det\left(\begin{bmatrix} 4 & 5-\lambda \\ 2 & 2 \end{bmatrix}\right) = 0$$

$$\iff (5-\lambda)\big((5-\lambda)(2-\lambda) - 4\big) - (4)\big(4(2-\lambda) - 4\big)$$

$$+ (2)\big(8 - 2(5-\lambda)\big) = 0$$

$$\overset{\text{CAS}}{\iff} (\lambda - 1)^2 (10 - \lambda) = 0$$

so $\lambda = 10$ or $\lambda = 1$ (with $\lambda = 1$ repeated).

E-vectors:

$\lambda = 10$:

$$\left[\begin{array}{ccc:c} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array}\right] \xRightarrow{\text{rref}} \left[\begin{array}{ccc:c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow v_1 = 2v_3, \quad v_2 = 2v_3$$

so $\vec{v} = t\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

One e-pair is $\{10, \{2, 2, 1\}\}$

$\lambda = 1$: $(A - \lambda I)\vec{v} = 0 \Rightarrow$

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \xRightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $v_1 + v_2 + \frac{1}{2} v_3 = 0$. Let $v_2 = s$, $v_3 = t$,

then

$$\vec{v} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

So there are two e-vectors corresponding to $\lambda = 1$

$$\{1, \{\{-1,1,0\}, \{-1,0,2\}\}\}$$

So the eigensystem is

$$\{\{10,1,1\}, \{\{2,2,1\}, \{-1,1,0\}, \{-1,0,2\}\}\}$$

Other thoughts

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A - \lambda I) = 0 \Rightarrow$ $(a-\lambda)(d-\lambda) - bc = 0 \Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$.

Now suppose that the e-values of A are λ_1 and λ_2.

Then $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \Rightarrow \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$.

That is $\lambda_1 + \lambda_2 = a + d = \text{Trace}(A)$ and $\lambda_1 \lambda_2 = ad - bc = \det(A)$.