Math 5360
Homework

1. Assignment One

Use a calculator or CAS for "exact" values.

(1) Compute the absolute error and relative error in the approximation of $p = \sqrt{2}$ by $\hat{p} = 1.414$.

(2) Compute the absolute error and relative error in the approximation of $p = 10^{\pi}$ by $\hat{p} = 1400$.

(3) Perform the following computation (i) exactly, (ii) using three-digit chopping arithmetic, and (iii) using three-digit rounding arithmetic. (iv) Compute the relative error in parts (ii) and (iii).

$$\left(\frac{1}{3} - \frac{3}{11}\right) + \frac{3}{20}$$

(4) Use three-digit rounding arithmetic to perform the following calculation. Compute the absolute error and relative error using a CAS accurate to at least five digits.

$$-10\pi + 6e - \frac{3}{62}$$

(5) Let $f(x) = \tan^{-1}(x)$.

(a) Find the first three non-zero terms in the Maclaurin series (Taylor series centered at $a = 0$) for $f(x)$.

(b) Calculate a theoretical bound on the error in using this Maclaurin polynomial in approximating f on the interval $(-0.5, 0.5)$.

(c) Compute the absolute error and relative error in the following approximation of π using the polynomial in the place of arctangent:

$$4\left[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right]$$

---

2. Assignment two

(1) Use three-digit chopping arithmetic to compute the following sum. For each part, which method is more accurate, and why?

$$\sum_{k=1}^{10} \frac{1}{k^3}$$

first by $\frac{1}{1} + \frac{1}{4} + ... + \frac{1}{1000}$ and then by $\frac{1}{1000} + \frac{1}{729} + ... + \frac{1}{1}$.

(2) The number e can be defined by $\sum_{n=0}^{\infty}(1/n!)$. Use four-digit chopping arithmetic to compute the following approximations to e and determine the absolute and relative errors.

(a) $e \approx \sum_{n=0}^{10}(1/n!)$

(b) $e \approx \sum_{j=0}^{10} 1/(10-j)!$

(3) Find the rate of convergence as $n \to \infty$.

(a) $\lim_{n\to\infty} \sin\left(\frac{1}{n^2}\right) = 0$

(b) $\lim_{n\to\infty} \left(\sin\left(\frac{1}{n}\right)\right)^2 = 0$

(4) Find the rate of convergence as $h \to 0$.

(a) $\lim_{h\to 0} \frac{1-\cos(h)}{h^2} = \frac{1}{2}$

(b) $\lim_{h\to 0} \frac{1+h-e^h}{h} = 0$

(5) Suppose that as x approaches zero,

$$F_1(x) = L_1 + O(x^a) \text{ and } F_2(x) = L_2 + O(x^b)$$

for constants a and b. Let c_1 and c_2 be nonzero constants and define

$$G(x) = c_1F_1(x) + c_2F_2(x) \text{ and } H(x) = F_1(c_1x) + F_2(c_2x).$$

Show that if $\mu = \text{minimum}(a, b)$, then as x approaches zero,

(a) $G(x) = c_1L_1 + c_2L_2 + O(x^\mu)$.

(b) $H(x) = L_1 + L_2 + O(x^\mu)$.

(6) Consider the sum $S = \sum_{i=1}^{n} \sum_{j=1}^{i} a_ib_j$.

(a) How many multiplications and additions are required to determine S?

(b) Modify the sum S to an equivalent form that reduces the number of computations.

(7) Construct an algorithm that has as input an integer $n \geq 1$, numbers x_0, x_1, ..., x_n, and a number x, and that produces as output the product $(x-x_0)(x-x_1)\cdots(x-x_n)$.

---

3. Assignment three

(1) For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

(a)

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -2x_1 - 4x_2 &= 6 \end{aligned}$$

(c)

$$\begin{aligned} 2x_1 + x_2 &= -1 \\ x_1 + x_2 &= 2 \\ x_1 - 3x_2 &= 5 \end{aligned}$$

(d)

$$2x_1 + x_2 + x_3 = 1$$
$$2x_1 + 4x_2 - x_3 = -1$$

(2) Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations. (The exact solution to each system is $x_1 = -1$, $x_2 = 1$, $x_3 = 3$.)

(a)

$$-x_1 + 4x_2 + x_3 = 8$$
$$\frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 = 1$$
$$2x_1 + x_2 + 4x_3 = 11$$

(b)

$$4x_1 + 2x_2 - x_3 = -5$$
$$\frac{1}{9}x_1 + \frac{1}{9}x_2 + \frac{1}{3}x_3 = -1$$
$$x_1 + 4x_2 + 2x_3 = 9$$

(3) Given the linear system

$$2x_1 - 6\alpha x_2 = 3$$
$$3\alpha x_1 - x_2 = \frac{3}{2}$$

(a) Find value(s) of α for which the system has no solutions.
(b) Find value(s) of α for which the system has an infinite number of solutions.
(c) Assuming a unique solution exist for a given α, find the solution.

---

4. Assignment four

Write (Matlab) programs that have as inputs the coefficient matrix and right-hand-side(s) of a system of linear equations and has as output the coefficient matrix transformed into upper-triangular form, the corresponding transformed rhs(s) and an index vector indicating any row interchanges performed. One program should implement partial pivoting, another should implement scaled partial pivoting. Then write a (Matlab) program that has as input the output from above and outputs the solution vector to the system of linear system of equations using back substitution.

(1) Test your programs on the following systems of linear equations.

(a)

$$x_1 + x_2 + x_3 + 5x_4 = -4.1$$
$$x_1 + x_2 + 2x_3 + 3x_4 = 1.7$$
$$2x_1 + 3x_2 - x_3 + 10x_4 = -23.5$$
$$3x_1 + 2 * x_2 + x_3 - 4 * x_4 = 5.9$$

(Solution: $x_1 = 2.3$, $x_2 = -4.5$, $x_3 = 3.6$, $x_4 = -1.1$.)

(b)

$$\begin{aligned} x_1 + .5x_2 + .33x_3 + .25x_4 + .2x_5 &= .99 \\ .5x_1 + .33x_2 + .25x_3 + .2x_4 + .17x_5 &= .64 \\ .33x_1 + .25x_2 + .2x_3 + .17x_4 + .14x_5 &= .45 \\ .25x_1 + .2x_2 + .17x_3 + .14x_4 + .13x_5 &= .45 \\ .2x_1 + .17x_2 + .14x_3 + .13x_4 + .11x_5 &= .31 \end{aligned}$$

(Solution: $x_1 = 1$, $x_2 = -2$, $x_3 = 3$, $x_4 = -4$, $x_5 = 5$.)

(c)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5.0 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 &= 6.6 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 &= 8.2 \\ 4x_1 + 5x_2 - 6x_3 + 7x_4 &= 3.8 \end{aligned}$$

(Solution: $x_1 = .1$, $x_2 = .3$, $x_3 = .5$, $x_4 = .7$.)

---

5. Assignment five

(1) Given the data $\{(4.0, 102.56), (4.2, 113.18), (4.5, 130.11), (4.7, 142.05), (5.1, 167.53), (5.5, 195.14), (5.9, 224.87), (6.3, 256.73), (6.8, 299.50), (7.1, 326.72)\}$,
   (a) Construct the least squares polynomial of degree 1 and compute the error.
   (b) Construct the least squares polynomial of degree 2 and compute the error.
   (c) Construct the least squares polynomial of degree 3 and compute the error.
   (d) Construct the least squares approximation of the form be^{ax} and compute the error.
   (e) Construct the least squares approximation of the form bx^a and compute the error.

(2) Construct the orthogonal polynomials that span the space of polynomials up to third degree on the intervals: (a) $[0, 1]$, (b) $[0, 2]$ and (c) $[1, 3]$.

(3) Find the linear least squares polynomial approximation (of degree 3) to $f(x)$ on the indicated interval if
   (a) $f(x) = \sin(2\pi x)$, $[0, 1]$;
   (b) $f(x) = e^x$, $[0, 2]$;
   (c) $f(x) = x\ln(x)$, $[1, 3]$:
   (d) $f(x) = \dfrac{1}{x+2}$, $[-1, 1]$.

---

6. Assignment six

(1) Determine the Padé approximation of degree 6 with $m = n = 3$ for $f(x) = \cos(x)$. Compare the results graphically with the Maclaurin polynomial of degree 6 (and $\cos(x)$) on the interval $[-\pi, \pi]$. Use a CAS to calculate the L_2 errors.

(2) Express the following rational function in continued-fraction form:

$$\frac{2x^3 + x^2 - x + 3}{3x^3 + 2x^2 - x + 1}$$

(3) Get the pseudo-code for the FFT algorithm from Dr. White and implement it in Matlab. Also write a routine that will except as inputs: the output of the FFT routine and a vector of x-values and will output the corresponding y-values of the trigonometric polynomial. Test your routines on $f(x) = 2x^2 - 9$ using 8 data points from the interval $[-\pi, \pi]$. Compare your results with the built in Matlab FFT routine.