MATH 3433
Lab07LineIntegrals

0. *Objective*

The objective of this project is to illustrate how line integrals in the plane can be computed using *Mathematica*. If you have not already done so, read the section of the text that introduces Line Integrals.

1. *Narrative*

Line integrals in the plane may be written in one of two forms. On one hand, if $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\vec{r} = \vec{r}(t) = (x(t), y(t))$, $t \in [a, b]$, is a parametrization of the curve C in the plane then

$$\int_C f\ ds = \int_{t=a}^{b} f(\vec{r}(t))|\vec{r}'(t)|\ dt.$$

On the other hand, if P and Q are any two functions that are continuous over C, then

$$\int_C P\ dx + Q\ dy = \int_{t=a}^{b} \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right)\ dt.$$

To see the connection between these forms, observe that if $\vec{F} = \langle P, Q\rangle$ is a vector field in the plane and $\vec{T} = \vec{r}'(t)/|\vec{r}'(t)|$ denotes the unit tangent vector field to C, then on one hand,

$$\int_C \vec{F}\cdot\vec{r} = \int_{t=a}^{b} \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\ dt = \int_{t=a}^{b} \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}'(t)}{|\vec{r}'(t)|}|\vec{r}'(t)|\ dt = \int_C \vec{F}\cdot\vec{T}\ ds = \int_C f\ ds.$$

On the other hand,

$$\int_{t=a}^{b} \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\ dt = \int_{t=a}^{b} \langle P, Q\rangle\cdot\langle\frac{dx}{dt}, \frac{dy}{dt}\rangle\ dt = \int_{t=a}^{b} P\ dx + Q\ dy.$$

Task

(1) Type the command lines below into *Mathematica* in the order in which they are listed; they illustrate how $\int_C f(x, y)\ ds$, where $f(x, y) = x^2 + 3y^2$ and C is the curve parametrized by $x = 2\cos(t)$, $y = 2\sin(t)$, $t \in [0, \pi]$, can be computed using *Mathematica*.

```
(* Project 16.2: Line Integrals in the Plane *)
(* Define the integrand *)
f[x_, y_] = x^2 + 3*y^2
(* Define the curve C: *)
r = {2*Cos[t], 2*Sin[t]}
(* Graph C *)
ParametricPlot[r, {t, 0, Pi}, PlotRange -> {{-2, 2}, {0, 2}},
   PlotStyle -> Blue]
(* Compute dr/dt *)
r1 = D[r, t]
(* Do the Line Integral *)
NIntegrate[f[r1[[1]], r1[[2]]]*Sqrt[r1[[1]]^2 + r1[[2]]^2],
   {t, 0, Pi}]
```

(2) Type the command lines below into *Mathematica* in the order in which they are listed; they illustrate how $\int_C \vec{F} \cdot d\vec{r}$ where C is the curve parametrized by $x = \cos(t)$, $y = \sin(t)$, $t \in [0, 2\pi]$, and $\vec{F}$ is the nonconservative field $\vec{F}(x, y) = \langle x - y, x - 2 \rangle$ can be computed suing *Mathematica*.

```
F[x_, y_] = {x - y, x - 2}
r = {Cos[t], Sin[t]}
field = VectorPlot[F[x, y], {x, -2, 2}, {y, -2, 2}];
curve = ParametricPlot[r, {t, 0, 2*Pi}, PlotStyle -> Red];
Show[field, curve]
dr = D[r, t]
integrand = Dot[F[r[[1]], r[[2]]], dr]
NIntegrate[integrand, {t, 0, 2*Pi}]
```

Notice that $\vec{F}$ is not conservative and the line integral is not zero. Indeed, the integral is twice the area of the unit circle, in agreement with Green's Theorem (which is covered in section 17.4).

(3) Repeat Task 2 letting C be the curve parametrized by $x = 2\cos(t)$, $y = 2\sin(t)$, $t \in [0, 2\pi]$ and $\vec{F}$ be the conservative field $\vec{F}(x, y) = \langle 2x, 6y \rangle = \nabla(x^2 + 3y^2)$. Observe that the line integral is (at least close to) zero, in agreement with The Fundamental Theorem for Line Integrals and Greens Theorem.

Note: The difference between a line integral of a conservative field $\vec{F}$ over a curve C and a line integral of a non-conservative field $\vec{F}$ over C is that in the conservative case, the projection of $\vec{F}$ along the unit tangent vector to C at each point is always balanced by the projection of $\vec{F}$ along the unit tangent vector to C at other points, but in the non-conservative case this need not happen.