

Name:
-------

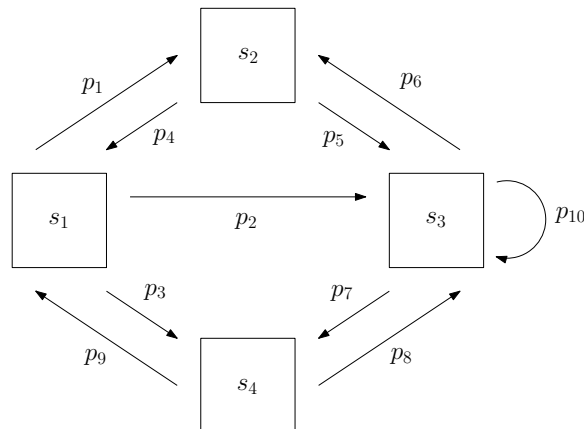
PIN:
------

## Applied Matrix Algebra Markov Chains

*Objective* The objective of this project is to illustrate how Markov Chains can be manipulated using Mathematica.

*Narrative* If you have not already done so, read Section 3.4 of the text. Markov chains are used as a way of modeling systems that sequentially change between several states. Examples of such systems include random walks, population migration, and simple gambling problems.

Simple models that use Markov Chains often use compartmental diagrams to show the relations between the various possible states. The compartmental diagram shown here represents a model that has four possible states (the boxes),  $\{s_1, s_2, s_3, s_4\}$ , and the possible change of states (the arrows) with their associated probabilities  $\{p_1, \dots, p_{10}\}$ .



In this diagram we see that an object that starts in state  $s_1$  can “move” to state  $s_2$  with probability  $p_1$ , or move to state  $s_3$  with probability  $p_2$ , or move to state  $s_4$  with probability  $p_3$ . Note that the object can not go directly from state  $s_2$  to  $s_4$  and that the object must move out of  $s_1$ ,  $s_2$ , or  $s_4$ , but may stay in  $s_3$  with probability  $p_{10}$ .

In this example we will assume that  $0 \leq p_i \leq 1$  and the sum of all outward probabilities for an individual state will be 1. i.e.  $p_4 + p_5 = 1$  and  $p_6 + p_7 + p_{10} = 1$ .

We can convert this compartmental diagram to a mathematical system of equations as follows. If  $\vec{x}_k$  is a vector that represents the  $k^{\text{th}}$  distribution of objects, then the  $k + 1$  distribution can be found through the matrix equation

$$\vec{x}_{k+1} = M\vec{x}_k$$

where  $M$  is called the transition matrix. In the above diagram the transition matrix is given by

$$M = \begin{bmatrix} 0 & p_4 & 0 & p_9 \\ p_1 & 0 & p_6 & 0 \\ p_2 & p_5 & p_{10} & p_8 \\ p_3 & 0 & p_7 & 0 \end{bmatrix}$$

Notice that in this transition matrix,  $m_{i,j}$  is the probability that an object currently in the  $j^{\text{th}}$  state will move to the  $i^{\text{th}}$  state in the next step.

Let's assign some probabilities to this diagram and use *Mathematica* to investigate this model. If  $p_1 = .4$ ,  $p_2 = .3$ ,  $p_3 = .3$ ,  $p_4 = .4$ ,  $p_5 = .5$ ,  $p_6 = .4$ ,  $p_7 = .4$ ,  $p_8 = .6$ ,  $p_9 = .4$ , and  $p_{10} = .2$ , then the transition matrix, in Maple, becomes

```
M = {{0, 1/2, 0, 2/5},
      {2/5, 0, 2/5, 0},
      {3/10, 1/2, 1/5, 3/5},
      {3/10, 0, 2/5, 0}};
```

```
MatrixForm[M]
```

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{2}{5} & 0 \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{5} & \frac{3}{5} \\ \frac{3}{10} & 0 & \frac{2}{5} & 0 \end{pmatrix}$$

Next suppose that the object starts in state  $s_1$ , then we use matrix multiplication to find the probability that the object will be in the state  $s_k$  by (using  $[1, 0, 0, 0]$  as the initial state)

```
x1 = {1, 0, 0, 0};
x2 = M.x1
```

$$\left\{0, \frac{2}{5}, \frac{3}{10}, \frac{3}{10}\right\}$$

This means, as we would expect, that there is a .4 probability that the object will be in state  $s_2$ , a .3 probability of being in state  $s_3$ , and a .3 probability of being in  $s_4$  after one transition.

To find the disposition of the object after two transitions we can do the following:

```
x3 = M.x2
```

$$\left\{\frac{8}{25}, \frac{3}{25}, \frac{11}{25}, \frac{3}{25}\right\}$$

```
N[x3, 8]
```

$$\{0.32000000, 0.12000000, 0.44000000, 0.12000000\}$$

Here the object has a 32% chance of ending up back in state  $s_1$ , a 12% chance of being in state  $s_2$ , a 44% chance of being in state  $s_3$ , and a 12% chance of being in state  $s_4$  after two transitions when starting at  $s_1 = \{1, 0, 0, 0\}$ .

What about after ten transitions? We could do this in a couple of ways. To answer this directly we could use exponentiation as follows

```
x11 = MatrixPower[M, 10, x1]
```

$$\left\{ \frac{4034527}{19531250}, \frac{2122472}{9765625}, \frac{2355859}{6250000}, \frac{31117757}{156250000} \right\}$$

```
N[x11, 8]
```

$$\{0.20656778, 0.21734113, 0.37693744, 0.19915364\}$$

Alternately, we could calculate all intermediary transitions:

```
s = Array[f, 11];
s[[1]] = x1;
Do[s[[k + 1]] = M.s[[k]], {k, 1, 10}];
```

Using this last method allows us to plot the various iterations.

```
plt1 = ListPlot[Table[{k, s[[k, 1]]}, {k, 1, 11}], PlotStyle -> Red,
  PlotMarkers -> {"\[Square]", 10},
  PlotRange -> {{0, 12}, {-0.2, 1.2}}];
plt2 = ListPlot[Table[{k, s[[k, 2]]}, {k, 1, 11}], PlotStyle -> Blue,
  PlotMarkers -> {"\[CircleDot]", 10},
  PlotRange -> {{0, 12}, {-0.2, 1.2}}];
plt3 = ListPlot[Table[{k, s[[k, 3]]}, {k, 1, 11}],
  PlotStyle -> Darker[Green, .5], PlotMarkers -> {"\[Diamond]", 18},
  PlotRange -> {{0, 12}, {-0.2, 1.2}}];
plt4 = ListPlot[Table[{k, s[[k, 4]]}, {k, 1, 11}],
  PlotStyle -> Darker[Orange], PlotMarkers -> {"\[Star]", 18},
  PlotRange -> {{0, 12}, {-0.2, 1.2}}];
Show[plt1, plt2, plt3, plt4]
```

Note that the output to these last commands is a graphics object that are not shown here. Also note that the symbols used as **PlotMarkers** can be found on the special symbol palette. For example `\[CircleDot]` should be the character  $\odot$  from the special character palette.

### Task

- (1) In the above example, what are the probabilities after 20 iterations, 100 iterations, and 1,000 iterations? What do you suppose will happen if we let the number of iterations go to infinity?
- (2) In the above example, start with one object in each of the states and calculate 1 transition, 10 transitions and 100 transitions. Interpret the results.
- (3) On page 159 of our book, do problems 6. Note: you will need to read about regular stochastic matrices to do parts (c) - (f).